$\frac{Q-41}{P-103}$. Evaluate $\oint_c \vec{F}\cdot d\vec{r}$ where $\vec{F} = (x-3y)\hat{i} + (y-2x)\hat{j}$ and c is the closed curve in the xy plane; $x = 2\cos t$; $y = 3\sin t$ from $t=0$ to $t=2\pi$.

A:- $\oint_c \vec{F}\cdot d\vec{r} = \oint_c \{(x-3y)\hat{i} + (y-2x)\hat{j}\}\cdot\{dx\hat{i} + dy\hat{j}\}$

$\vec{r} = x\hat{i} + y\hat{j}$
$d\vec{r} = dx\hat{i} + dy\hat{j}$

$x = 2\cos t$
$dx = -2\sin t\, dt$
& $y = 3\sin t$
$dy = 3\cos t\, dt$

$= \oint_c (x-3y)\,dx + (y-2x)\,dy$.

$= \int_{t=0}^{2\pi} (2\cos t - 9\sin t)(-2\sin t\, dt) + (3\sin t - 4\cos t)(3\cos t)\, dt$

$= \int_{t=0}^{2\pi} (-4\sin t\cos t + 18\sin^2 t + 9\sin t\cos t - 12\cos^2 t)\, dt$

$= \int_{t=0}^{2\pi} (5\sin t\cos t + 18\sin^2 t - 12\cos^2 t)\, dt$

$= \int_0^{2\pi} \{\frac{5}{2}\sin 2t + 9(1-\cos 2t) - 6(1+\cos 2t)\}\, dt$

$= \left[\frac{5}{2}\cdot -\frac{\cos 2t}{2} + 9t - 9\cdot\frac{\sin 2t}{2} - 6t - \cancel{6}^{3}\cdot\frac{\sin 2t}{\cancel{2}}\right]_0^{2\pi}$

$= \left[-\frac{5}{4}\cos 2t + 3t - \frac{15}{2}\sin 2t\right]_0^{2\pi}$

$= \left(-\frac{5}{4}\cos 4\pi + 6\pi - \frac{15}{2}\sin 4\pi\right) - \left(-\frac{5}{4}\cdot 1 + 0 - 0\right)$

$= -\frac{5}{4} + 6\pi - \frac{15}{2}\cdot 0 + \frac{5}{4}$

$= 6\pi$; if c traversed in positive (counter clockwise) direction.

N.B: $\int_c \vec{F}\cdot d\vec{r}$ is the work done by $\vec{F}$ ~~along~~ Ans the curve c.

37 (P-102) / SOS. If $\vec{A} = (2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}$, Evaluate $\int_c \underline{A} \cdot d\underline{r}$ along the path c:

(a) $x = 2t^2,\ y = t,\ z = t^3$ from $t=0$ to $t=1$.

(b) The st. lines from $(0,0,0)$ to $(0,0,1)$, then to $(0,1,1)$ and then to $(2,1,1)$.

(c) the st. line joining $(0,0,0)$ to $(2,1,1)$.

A:- $$\int_c \underline{A} \cdot d\underline{r} = \int_c [(2y+3)\hat{i} + xz\hat{j} + (yz-x)\hat{k}] \cdot [dx\hat{i} + dy\hat{j} + dz\hat{k}]$$

$$= \int_c (2y+3)\,dx + xz\,dy + (yz-x)\,dz.$$

(a) $x = 2t^2,\ y = t,\ z = t^3$ ~~from points (0,0,0) and (0,0,1) correspond~~ from $t=0$ ~~to~~ and $t=1$ ~~respectively~~.

then $$\int \underline{A} \cdot d\underline{r} = \int_{t=0}^{1} (2t+3)(4t)\,dt + (2t^5)(t^3)\,dt + (t^4 - 2t^2)(3t^2)\,dt$$

$$= \int_{t=0}^{1} (8t^2 + 12t)\,dt + 2t^5\,dt + (3t^6 - 6t^4)\,dt$$

$$= \left[\frac{8t^3}{3} + 6t^2 + \frac{1}{3}t^6 + \frac{3}{7}t^7 - \frac{6}{5}t^5 \right]_0^1$$

$$= \frac{8}{3} + 6 + \frac{1}{3} + \frac{3}{7} - \frac{6}{5}$$

$$= \frac{280 + 630 + 35 + 45 - 126}{105}$$

$$= \frac{\cancel{864}^{288}}{\cancel{105}_{35}} = \frac{288}{35}$$ Ans.

(b) Along the st. line from $(0,0,0)$ to $(0,0,1)$, $x=0,\ y=0$

i.e $dx = 0$, $dy = 0$ while z varies from 0 to 1.

given the integral over this part is

$$\int_{z=0}^{1} (0+3)(0) + (0)(0) + (0-0)\,dz = 0$$

Along the st. line from $(0,0,1)$ to $(0,1,1)$;

$x=0$, y varies from 0 to 1 & $z=1$ $\therefore dx=0,\ dz=0$

$\therefore$ the integral over this part is

$$\int_{y=0}^{1} (2y+3)(0) + (0)(1)\,dy + (y-0)(0) = 0$$

$= \cancel{\left[\frac{y^2}{2}\right]_0^1} \cancel{=}$ Along the st. line from $(0,1,1)$ to $(2,1,1)$

x varies from 0 to 2, $y=1$, $z=1$

$\therefore dy=0,\ dz=0$

$\therefore$ the integral over this part is

$$\int_{x=0}^{2} (2+3)\,dx + (x)(1)(0) + \{(1)(1) - x\}(0)$$

$$= \cancel{\left[5x\right]_0^2} = \int_{x=0}^{2} 5\,dx + 0 + 0 = \left[5x\right]_0^2 = 10$$

Adding $\int_C \underline{A} \cdot \underline{dr} = 0 + 0 + 10 = 10$ Ans.

47 (P-103)
SOS

(a) Prove that $\vec{F} = (y^2\cos x + z^3)\hat{i} + (2y\sin x - 4)\hat{j} + (3xz^2 + 2)\hat{k}$ is a conservative force field.

(b) Find the scalar potential for $\vec{F}$.

(c) Find the work done in moving an object in this field from $(0,1,-1)$ to $(\frac{\pi}{2}, -1, 2)$.

A:- If $\underline{\nabla} \times \underline{F} = 0$, then $\vec{F}$ is a conservative force field.

Now $$\underline{\nabla} \times \underline{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ y^2\cos x + z^3 & 2y\sin x - 4 & 3xz^2 + 2 \end{vmatrix}$$

$$= \left[\frac{\partial}{\partial y}(3xz^2+2) - \frac{\partial}{\partial z}(2y\sin x - 4)\right]\hat{i} - \left[\frac{\partial}{\partial x}(3xz^2+2) - \frac{\partial}{\partial z}(y^2\cos x + z^3)\right]\hat{j} + \left[\frac{\partial}{\partial x}(2y\sin x - 4) - \frac{\partial}{\partial y}(y^2\cos x + z^3)\right]\hat{k}$$

$$= 0 - (3z^2 - 3z^2)\hat{j} + (2y\cos x - 2y\cos x)\hat{k}$$

$$= 0$$

$\therefore \vec{F}$ is a conservative force field.

(b) Since $\vec{F}$ is a conservative force field, so $\underline{\nabla} \times \underline{F} = 0$
i.e $\vec{F} = \underline{\nabla}\phi$ (because $\underline{\nabla} \times \underline{\nabla}\phi = 0$)

Now $$\underline{F} \cdot \underline{dr} = \underline{\nabla}\phi \cdot \underline{dr} = \frac{\partial \phi}{\partial x}dx + \frac{\partial \phi}{\partial y}dy + \frac{\partial \phi}{\partial z}dz$$
$$= d\phi$$

$$\therefore d\phi = \underline{F} \cdot \underline{dr} = (y^2\cos x + z^3)dx + (2y\sin x - 4)dy + (3xz^2 + 2)dz$$

$$= (y^2\cos x\,dx + 2y\sin x\,dy) + (z^3dx + 3xz^2dz) - 4dy + 2dz$$

$$= d(y^2\sin x) + d(xz^3) - 4dy + 2dz$$

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$$d\phi = d(y^2\sin x + xz^3) - 4\,dy + 2\,dz$$

Integrating; $\phi = y^2\sin x + xz^3 - 4y + 2z + \text{constant}$. Ans

(e) work done $= \int_{P_1}^{P_2} \underline{F}\cdot d\underline{r}$

$$= \int_{P_1}^{P_2} (y^2\cos x + z^3)\,dx + (2y\sin x - 4)\,dy + (3xz^2 + 2)\,dz$$

$$= \int_{P_1}^{P_2} \left[d(y^2\sin x + xz^3) - 4\,dy + 2\,dz\right]$$

$$= y^2\sin x + xz^3 - 4y + 2z \Big|_{P_1}^{P_2}$$

$$= y^2\sin x + xz^3 - 4y + 2z \Big|_{(0,1,-1)}^{(\frac{\pi}{2},-1,2)}$$

$$= \left(1 + \frac{\pi}{2}\cdot 8 + 4 + 4\right) - (0 + 0 - 4 - 2)$$

$$= 9 + 4\pi + 6$$

$$= 15 + 4\pi \cdot \text{ Ans}$$